

OTHER QUANTUM RELATIVES OF THE ALEXANDER POLYNOMIAL THROUGH THE LINKS-GOULD INVARIANTS

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ABSTRACT. Oleg Viro studied in [14] two interpretations of the (multivariable) Alexander polynomial as a quantum link invariant: either by considering the quasi triangular Hopf algebra associated to $U_q\mathfrak{sl}(2)$ at fourth roots of unity, or by considering the super Hopf algebra $U_q\mathfrak{gl}(1|1)$. In this paper, we show these Hopf algebras share properties with the -1 specialization of $U_q\mathfrak{gl}(n|1)$ leading to the proof of a conjecture of David De Wit, Atsushi Ishii and Jon Links on the Links-Gould invariants.

1. INTRODUCTION

The Links-Gould invariants of links $LG^{n,m}$ are two variable quantum link invariants. They are derived from super Hopf Algebras $U_q\mathfrak{gl}(n|m)$. David De Wit, Atsushi Ishii and Jon Links conjectured [3] that for any link L

$$LG^{n,m}(L; \tau, e^{i\pi/m}) = \Delta_L(\tau^{2m})^n,$$

where Δ_L is the Alexander-Conway polynomial of L . They proved the conjecture when $(n, m) = (1, m)$ and when $(n, m) = (2, 1)$ for a particular class of braids. A complete proof of the $(n, 1)$ case for $n = 2, 3$ is given in [8]. However this is achieved by studying the invariants at hand at the level of *representations*, which requires computation of an explicit R -matrix for each $LG^{n,1}$, making that method hard to implement as n grows.

Here we prove the $(n, 1)$ case of the conjecture for any n :

$$LG^{n,1}(L; \tau, -1) = \Delta_L(\tau^2)^n.$$

To do so we study the structure of the *universal* objects directly, and in particular the (super) Hopf algebras and universal R -matrices that are involved.

However, the strong version of the conjecture is still open.

2. HOPF ALGEBRAS FOR THE ALEXANDER POLYNOMIAL

We first define a Hopf algebra U which is an essential ingredient for the quantum relatives of the Alexander polynomial. Unfortunately this algebra is only braided in a weak sense. Then we recall two quantum groups which can be seen as central extensions of U . One was first used by Murakami [10], both were studied by Viro in [14]. Finally we compare the braidings of the two Hopf algebras.

2.1. A braided Hopf algebra U . The following Hopf algebra U is a version of quantum $\mathfrak{sl}(2)$ when the quantum parameter q is a fourth root i of 1. The complex algebra U is presented by generators $k^{\pm 1}, e, f$ and relations

$$ke + ek = kf + fk = e^2 = f^2 = 0 \quad \text{and} \quad ef - fe = k - k^{-1}.$$

The coproduct, counity and antipode of U are given by

$$\begin{aligned}\Delta(e) &= 1 \otimes e + e \otimes k, & \varepsilon(e) &= 0, & S(e) &= -ek^{-1}, \\ \Delta(f) &= k^{-1} \otimes f + f \otimes 1, & \varepsilon(f) &= 0, & S(f) &= -kf, \\ \Delta(k) &= k \otimes k, & \varepsilon(k) &= 1, & S(k) &= k^{-1}.\end{aligned}$$

This Hopf algebra can be seen in a sense as a a "double" of Bodo Pareigis' Hopf algebra [11] that would be $\langle k, f \rangle$ with our notations. A pivotal structure is a group like element ϕ whose conjugation is equal to the square of the antipode. There is non obviously a better choice which is given by $\phi = k^{-1}$.

Let $\tau : x \otimes y \mapsto y \otimes x$ be the switch of factors. Hopf algebra U is not quasi-triangular but it is braided in the sense of [12]: there exists an (outer) algebra automorphism $\mathcal{R} : U \otimes U \rightarrow U \otimes U$ different from τ that satisfies

$$\begin{aligned}(1) \quad & \mathcal{R} \circ \Delta = \tau \circ \Delta, \\ (2) \quad & \Delta_1 \circ \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}, \\ (3) \quad & \Delta_2 \circ \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}.\end{aligned}$$

Automorphism \mathcal{R} admits a regular splitting (see [12]) $\mathcal{R} = \mathcal{D} \circ \text{Ad}_{\tilde{R}}$ where $\text{Ad}_{\tilde{R}}$ is the conjugation by the invertible element

$$\tilde{R} = 1 + e \otimes f$$

and \mathcal{D} is an outer automorphism satisfying equations similar to (2) and (3) and defined by:

$$\mathcal{D} \circ \tau = \tau \circ \mathcal{D}, \quad \mathcal{D}(e \otimes 1) = e \otimes k, \quad \mathcal{D}(f \otimes 1) = f \otimes k^{-1} \quad \text{and} \quad \mathcal{D}(k \otimes 1) = k \otimes 1.$$

The elements $k^{\pm 2}$ generate a central sub-Hopf algebra and for any $g \in \mathbb{C} \setminus \{0, 1\}$, the quotient $U/(k^2 - g)$ is a 8-dimensional semi-simple Hopf algebra with two isomorphism classes of irreducible representations $V_{\pm a}$ where $a^2 = g$. The representation V_a is 2-dimensional and can be written in a certain basis (e_0, e_1)

$$(4) \quad k = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ a - \frac{1}{a} & 0 \end{pmatrix}.$$

The central element $ef + fe$ acts by $(a - a^{-1})I_2$.

2.2. The $\mathfrak{sl}(2)$ model and the Alexander polynomial. From [14, 1] the $\mathfrak{sl}(2)$ model is the unrolled version of quantum $\mathfrak{sl}(2)$ at $q = \mathbf{i} = \exp(\mathbf{i}\pi/2)$. It is an algebra $U_{\mathbf{i}}^H \mathfrak{sl}(2)$ generated by $K^{\pm 1}, E, F, H$. Its presentation is obtained from that of U ($U_0 = \langle K^{\pm 1}, E, F \rangle \simeq U$) by adding the generator H and the following relations:

$$[H, K] = 0, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

We will consider the category \mathcal{C} of weight modules: finite dimensional vector spaces where element H acts diagonally and

$$(5) \quad K = \mathbf{i}^H = \exp(\mathbf{i}\pi H/2).$$

The pivotal Hopf algebra structure U is extended to $U_{\mathbf{i}}^H \mathfrak{sl}(2)$ by the following relations: ¹

$$\Delta(H) = 1 \otimes H + H \otimes 1, \quad \varepsilon(H) = 0, \quad S(H) = -H.$$

¹Compared to Viro, we use the opposite coproduct here.

As in U , the pivotal element is $\Phi = K^{-1}$ so that $S^2(\cdot) = \Phi \cdot \Phi^{-1}$. With this pivotal structure, category \mathcal{C} is ribbon with braiding given by the switch $\tau : x \otimes y \mapsto y \otimes x$ composed with the action of the universal R -matrix:

$$R^H = \mathbf{i}^{H \otimes H/2} (1 + E \otimes F).$$

Lemma 2.1. *For any two representations $V, W \in \mathcal{C}$, the conjugation in $V \otimes W$ by $D^H := \mathbf{i}^{H \otimes H/2}$ induces an automorphism \mathcal{D}^H of $\text{End}_{\mathbb{C}}(V \otimes W)$ which satisfies*

$$\rho_{V \otimes W} \circ \mathcal{D} = \mathcal{D}^H \circ \rho_{V \otimes W} : U \otimes U \rightarrow \text{End}_{\mathbb{C}}(V \otimes W).$$

Proof. This is an easy consequence of Equation (5). More generally, if $x, y \in U$ satisfy $[H, x] = 2mx$ and $[H, y] = 2m'y$, then $H \otimes H \cdot x \otimes y = x \otimes y \cdot (H + 2m) \otimes (H + 2m')$ so

$$\begin{aligned} \mathbf{i}^{\rho_{V \otimes W}(H \otimes H/2)} \rho_{V \otimes W}(x \otimes y) &= \rho_{V \otimes W}(x \otimes y) \mathbf{i}^{\rho_{V \otimes W}((H+2m) \otimes (H+2m')/2)} \\ &= \rho_{V \otimes W}((x \otimes K^m)(K^{m'} \otimes y)) \mathbf{i}^{\rho_{V \otimes W}(H \otimes H/2)} \\ &= \rho_{V \otimes W}(\mathcal{D}(x \otimes y)) \mathbf{i}^{\rho_{V \otimes W}(H \otimes H/2)}. \end{aligned} \quad \boxed{2.1}$$

For each complex number α which is not an odd integer, $U_{\mathbf{i}}^H \mathfrak{sl}(2)$ possesses, up to isomorphism, a unique two dimensional irreducible representation V_{α} with $\text{Spec}(H) = \{\alpha + 1, \alpha - 1\}$. Its restriction to U is representation V_a where $a = \mathbf{i}^{\alpha+1}$ and the action of H is given by $H = \begin{pmatrix} \alpha + 1 & 0 \\ 0 & \alpha - 1 \end{pmatrix}$.

In the representation $V_{\alpha} \otimes V_{\beta}$, with respect to basis $(e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1)$ the braiding is:

$$\mathbf{i}^{\frac{\alpha\beta-1}{2}} \begin{pmatrix} \mathbf{i}^{\frac{\alpha+\beta+2}{2}} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{i}^{\frac{\alpha-\beta}{2}} & 0 \\ 0 & \mathbf{i}^{\frac{-\alpha+\beta}{2}} & \mathbf{i}^{\frac{-\alpha+\beta}{2}} (\mathbf{i}^{\beta+1} - \mathbf{i}^{-\beta-1}) & 0 \\ 0 & 0 & 0 & \mathbf{i}^{\frac{-\alpha-\beta+2}{2}} \end{pmatrix}.$$

In the case where $\alpha = \beta$, the R -matrix then takes the particular form

$$\tau R^H = \mathbf{i}^{\frac{\alpha^2-1}{2}} \begin{pmatrix} t^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & (t^{-1/2} - t^{1/2}) & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{pmatrix}$$

where we set $t^{1/2} = \mathbf{i}^{-\alpha-1}$.

The ribbon category we consider here allows us to apply the Reshetikhin-Turaev theory [13] to construct a framed link isotopy invariant in S^3 . It becomes an unframed link isotopy invariant if one divides the above R -matrix on $V_{\alpha} \otimes V_{\alpha}$ by the value of the twist $\theta_{\alpha} = \mathbf{i}^{\frac{\alpha^2-1}{2}}$. In this particular case, the invariant we find is the Conway normalization of the classical Alexander polynomial, see [14]. The Links-Gould invariants $LG^{n,1}$ that will interest us in the following are obtained by the same general construction using other Hopf algebras. Explicitly, the Reshetikhin-Turaev functor gives representations of braid groups B_{ℓ}

$$\begin{aligned} \Psi_{V_{\alpha}^{\otimes \ell}} : B_{\ell} &\longrightarrow GL(V_{\alpha}^{\otimes \ell}) \\ \sigma_i &\mapsto \text{Id}_{V_{\alpha}^{\otimes i-1}} \otimes \theta_{\alpha}^{-1} \tau R^H \otimes \text{Id}_{V_{\alpha}^{\otimes \ell-i-1}}, \end{aligned}$$

where σ_i is the i^{th} standard Artin generator of braid group B_{ℓ} .

Let L be an oriented link in S^3 obtained as closure of a braid in ℓ strands $b \in B_{\ell}$. Then:

- 1) There exists a scalar c such that $\text{trace}_{2,3,\dots,\ell}((\text{Id}_{V_{\alpha}} \otimes (K^{-1})^{\otimes \ell-1}) \circ \Psi_{V_{\alpha}^{\otimes \ell}}(b)) = c \cdot \text{Id}_{V_{\alpha}}$,
- 2) $L \mapsto c$ is a link invariant and is equal to the Alexander polynomial of L , $\Delta_L(t)$.

Remark 2.2. Identifying algebras $\text{End}_{\mathbb{C}}(V_{\alpha}^{\otimes \ell})$ and $\text{End}_{\mathbb{C}}(V_{\alpha})^{\otimes \ell}$, the partial trace operator is defined by $\text{trace}_{2,3,\dots,\ell}(f_1 \otimes \dots \otimes f_{\ell}) := \text{trace}(f_2) \text{trace}(f_3) \dots \text{trace}(f_{\ell}) f_1 \in \text{End}_{\mathbb{C}}(V_{\alpha})$ for any $f_1, \dots, f_{\ell} \in \text{End}_{\mathbb{C}}(V_{\alpha})$.

2.3. An example of bosonization: the $\mathfrak{gl}(1|1)$ model.

2.3.1. *Bosonization.* Here we recall Majid's trick [9] to transform a super Hopf algebra into an ordinary one.

Let H be a pivotal super Hopf algebra and \mathcal{C} be its even monoidal category of representations (morphisms are formed by even H -linear maps). Let H^{σ} be the bosonization of H : as an algebra, H^{σ} is the semi-direct product of H with $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ where the action of σ or equivalently the commutation relations in H^{σ} are given by

$$\forall x \in H, \sigma x = (-1)^{|x|} x \sigma.$$

The coproduct Δ^{σ} on H^{σ} is given by $\Delta^{\sigma} \sigma = \sigma \otimes \sigma$ and

$$\forall x \in H, \Delta^{\sigma}(x) = \sum_i x_i \sigma^{|x'_i|} \otimes x'_i \text{ where } \Delta(x) = \sum_i x_i \otimes x'_i.$$

If $R = \sum_i R_i^{(1)} \otimes R_i^{(2)}$ is the universal R -matrix in H , then the following formula defines a universal R -matrix in H^{σ} :

$$R^{\sigma} = R_1 \sum_i R_i^{(1)} \sigma^{|R_i^{(2)}|} \otimes R_i^{(2)}, \text{ where } R_1 = \frac{1}{2}(1 \otimes 1 + \sigma \otimes 1 + 1 \otimes \sigma - \sigma \otimes \sigma).$$

Given a super representation $V = V_{\overline{0}} \oplus V_{\overline{1}}$ of H we get a representation of H^{σ} by setting $\sigma|_V = \text{Id}_{V_{\overline{0}}} - \text{Id}_{V_{\overline{1}}}$. Reciprocally, since $\sigma^2 = 1$, every H^{σ} -module inherits a natural $\mathbb{Z}/2\mathbb{Z}$ grading: W splits into $W = W_{\overline{0}} \oplus W_{\overline{1}}$ where we define $W_{\overline{0}} = \ker(\sigma - 1)$ and $W_{\overline{1}} = \ker(\sigma + 1)$.

Theorem 2.3 (Majid Theorem 4.2). *The even category of super H -modules can be identified with the category of H^{σ} -modules.*

Remark that the antipode of H^{σ} is given by $x \mapsto \sigma^{|x|} S(x)$ and if H as a pivot ϕ then one can choose $\phi^{\sigma} = \sigma \phi$ as a pivot in H^{σ} .

2.3.2. *The $\mathfrak{gl}(1|1)$ model.* Using the same notations as Viro: $U_q \mathfrak{gl}(1|1)$ is the pivotal super Hopf algebra generated by two odd generators X, Y , two even generators I, G satisfying the relations

$$\begin{aligned} XY + YX &= \frac{C - C^{-1}}{q - q^{-1}}, \quad X^2 = Y^2 = 0, \\ [I, X] &= [I, Y] = [I, G] = 0, \\ [G, X] &= X, \quad [G, Y] = -Y, \end{aligned}$$

where $C = q^I$, with coproduct

$$\begin{aligned} \Delta(I) &= 1 \otimes I + I \otimes 1, \quad \Delta(G) = 1 \otimes G + G \otimes 1, \\ \Delta(X) &= X \otimes C^{-1} + 1 \otimes X, \quad \Delta(Y) = Y \otimes 1 + C \otimes Y, \end{aligned}$$

countit

$$\varepsilon(X) = \varepsilon(Y) = \varepsilon(I) = \varepsilon(G) = 0,$$

antipode

$$S(I) = -I, \quad S(G) = -G, \quad S(X) = -XC, \quad S(Y) = -YC^{-1},$$

pivot

$$\phi = K$$

and universal R -matrix

$$R = (1 + (q - q^{-1})(X \otimes Y)(C \otimes C^{-1}))q^{-I \otimes G - G \otimes I}.$$

Its bosonization $U_q \mathfrak{gl}(1|1)^\sigma$ contains a sub-Hopf algebra U_1 isomorphic to U given by

$$e = (q - q^{-1})X\sigma, \quad f = Y \quad \text{and} \quad k = C^{-1}\sigma.$$

Indeed, these elements satisfy the following:

$$ef - fe = (q - q^{-1})(X\sigma Y - YX\sigma) = (q - q^{-1})(-XY - YX)\sigma = k - k^{-1},$$

$$ke + ek = kf + fk = 0,$$

$$\Delta^\sigma(e) = (q - q^{-1})\Delta^\sigma(X\sigma) = (q - q^{-1})(X \otimes C^{-1} + \sigma \otimes X)(\sigma \otimes \sigma) = e \otimes k + 1 \otimes e,$$

$$\Delta^\sigma(f) = \Delta^\sigma(Y) = Y \otimes 1 + C\sigma \otimes Y = f \otimes 1 + k^{-1} \otimes f,$$

$$\Delta^\sigma(k) = k \otimes k.$$

In the bosonization, the universal R -matrix is

$$R^\sigma = R_1 q^{-(I \otimes G + G \otimes I)}(1 + e \otimes f), \text{ where } R_1 = \frac{1}{2}(1 \otimes 1 + \sigma \otimes 1 + 1 \otimes \sigma - \sigma \otimes \sigma).$$

Lemma 2.4. Denoting $D' = q^{-I \otimes G - G \otimes I}$ and $D^\sigma = R_1 D'$ we have, for any $x, y \in U = U_1$:

$$R_1(x \otimes y)R_1^{-1} = \sigma^{|y|}x \otimes y \sigma^{|x|}, \quad D'(x \otimes y)(D')^{-1} = xC^{-d_G(y)} \otimes yC^{-d_G(x)},$$

$$D^\sigma(x \otimes y)(D^\sigma)^{-1} = (C^{-1}\sigma)^{d_G(y)}x \otimes y(C^{-1}\sigma)^{d_G(x)} = \mathcal{D}(x \otimes y),$$

where $d_G(x) \in \mathbb{Z}$ is defined by $[G, x] = d_G(x)x$.

Remark 2.5. For a homogeneous $a \in U_0$, $|a| = d_G(a)$ modulo 2.

Let us recall a family of 2-dimensional $U_q \mathfrak{gl}(1|1)^\sigma$ -modules. This family is parametrized by two complex numbers (j, J) and $\varepsilon \in \{0, 1\}$, see [14]. It extends the representation V_a of U_1 where $a = (-1)^\varepsilon q^{-2j}$. Written in matrix form,

$$I = \begin{pmatrix} 2j & 0 \\ 0 & 2j \end{pmatrix}, G = \begin{pmatrix} \frac{J+1}{2} & 0 \\ 0 & \frac{J-1}{2} \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & \frac{q^{2j} - q^{-2j}}{q - q^{-1}} \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma = \begin{pmatrix} (-1)^\varepsilon & 0 \\ 0 & -(-1)^\varepsilon \end{pmatrix}.$$

2.4. Comparing the actions of R^σ and R^H . $U_0 \subset U_{\mathbf{i}}^H \mathfrak{sl}(2)$ and $U_1 \subset U_q \mathfrak{gl}(1|1)^\sigma$ are two isomorphic Hopf algebras. The goal of this paragraph is to show the action of

$$R^H = \mathbf{i}^{H \otimes H/2}(1 + E \otimes F) \in U_{\mathbf{i}}^H \mathfrak{sl}(2) \otimes U_{\mathbf{i}}^H \mathfrak{sl}(2)$$

and that of

$$R^\sigma = R_1 q^{-(I \otimes G + G \otimes I)}(1 + e \otimes f) \in U_q \mathfrak{gl}(1|1)^\sigma \otimes U_q \mathfrak{gl}(1|1)^\sigma$$

on two representations $V_1^H \otimes V_2^H$ of $U_{\mathbf{i}}^H \mathfrak{sl}(2)$ and $V_1^\sigma \otimes V_2^\sigma$ of $U_q \mathfrak{gl}(1|1)^\sigma$ are identical up to a scalar multiple of the identity, when V_i^H and V_i^σ have the same underlying $U_0 = U_1$ -module structure.

We recall conjugations by the elements D^H on one side and D^σ on the other side both induce the same automorphism \mathcal{D} of $U \otimes U$.

Proposition 2.6. Set for $i = 1, 2$ V_i^H a representation of $U_{\mathbf{i}}^H \mathfrak{sl}(2)$ and V_i^σ a representation of $U_q \mathfrak{gl}(1|1)^\sigma$ which both restrict to the same irreducible representation of $U = U_0 = U_1$. Then $D^H(D^\sigma)^{-1} \in \text{End}_{\mathbb{C}}(V_1 \otimes V_2)$ is a scalar multiple of the identity.

Proof. The density theorem states that if V is a finite dimensional irreducible representation of an algebra A over an algebraically closed field, then $A \twoheadrightarrow \text{End}(V)$ is surjective. Denote the representations at hand $\rho_{V_i^H}, \rho_{V_i^\sigma}$ for $i = 1, 2$. We supposed

$$\rho_{V_i^H}|_U = \rho_{V_i^\sigma}|_U.$$

So if $\rho_H = \rho_{V_1^H} \otimes \rho_{V_2^H}$ and $\rho_\sigma = \rho_{V_1^\sigma} \otimes \rho_{V_2^\sigma}$ we define $\rho := \rho_H|_{U \otimes U} = \rho_\sigma|_{U \otimes U}$. Using Lemma 2.1 and Lemma 2.4, for any $x, y \in U$:

$$\rho_H(D^H)\rho(x \otimes y)\rho_H((D^H)^{-1}) = \rho(\mathcal{D}(x \otimes y)) = \rho_\sigma(D^\sigma)\rho(x \otimes y)\rho_\sigma((D^\sigma)^{-1}).$$

Which means

$$\rho_H(D^H)^{-1}\rho_\sigma(D^\sigma)\rho(x \otimes y) = \rho(x \otimes y)\rho_H(D^H)^{-1}\rho_\sigma(D^\sigma).$$

Using the density theorem, $\rho_H(D^H)^{-1}\rho_\sigma(D^\sigma)$ commutes with any element in $\text{End}_{\mathbb{C}}(V_1) \otimes \text{End}_{\mathbb{C}}(V_2) = \text{End}_{\mathbb{C}}(V_1 \otimes V_2)$. So this linear map is a scalar multiple of the identity. 2.6

From now on, we consider Hopf algebra $A = U_{\mathbf{i}}^H \mathfrak{sl}(2) \otimes_U U_q \mathfrak{gl}(1|1)^\sigma$. A contains both algebras $U_{\mathbf{i}}^H \mathfrak{sl}(2)$ and $U_q \mathfrak{gl}(1|1)^\sigma$.

Formally, setting $q = e^h$, $q^T := e^{h^T}$ and $\mathbf{i}^\alpha = e^{\mathbf{i}\frac{\pi}{2}\alpha}$, we also consider that

$$\mathbf{i}^H = k = q^{-I}\sigma$$

which means that we will only study representations of A that satisfy this relation. Recall from Equations (4) the representation of U with parameter a . We can look for the representations of A that simultaneously extend to the representations of $U_{\mathbf{i}}^H \mathfrak{sl}(2)$ and $U_q \mathfrak{gl}(1|1)^\sigma$ we already described. If $\varepsilon \in \{0, 1\}$ is the degree of the first vector e_0 of the basis (e_0, e_1) we choose, direct computation of such a representation $V(\alpha, a, 2j, \varepsilon, J)$ shows it is well defined if and only if:

$$(6) \quad \begin{cases} (-1)^\varepsilon q^{-2j} = a \\ a = e^{\mathbf{i}\frac{\pi}{2}(\alpha+1)} = \mathbf{i}^{\alpha+1} \end{cases}$$

Setting $s = q^j \mathbf{i}^{\frac{\alpha-3-2\varepsilon}{2}} = \pm 1$, we can compute the coefficient $R^H/R^\sigma = D^H/D^\sigma$ given by Proposition 2.6 in our case.

Proposition 2.7. $R^H/R^\sigma = D^H/D^\sigma = ss'(-1)^{\varepsilon\varepsilon'} \mathbf{i}^{\varepsilon+\varepsilon'} \mathbf{i}^{\frac{\alpha\alpha'-1}{2}} q^{jJ'+j'J}$.

Proof. Using representation $V \otimes V' = V(\alpha, a, 2j, \varepsilon, J) \otimes V(\alpha', a', 2j', \varepsilon', J')$ in basis $(e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1)$, we can write:

$$D^H = \mathbf{i}^{\alpha\alpha'/2} \begin{pmatrix} \mathbf{i}^{\frac{\alpha+\alpha'+1}{2}} & 0 & 0 & 0 \\ 0 & \mathbf{i}^{\frac{-\alpha+\alpha'-1}{2}} & 0 & 0 \\ 0 & 0 & \mathbf{i}^{\frac{\alpha-\alpha'-1}{2}} & 0 \\ 0 & 0 & 0 & \mathbf{i}^{\frac{-\alpha-\alpha'+1}{2}} \end{pmatrix}.$$

Moreover, $D^\sigma = R_1 D'$ and

$$R_1 = (-1)^{\varepsilon\varepsilon'} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1)^\varepsilon & 0 & 0 \\ 0 & 0 & (-1)^{\varepsilon'} & 0 \\ 0 & 0 & 0 & (-1)^{\varepsilon+\varepsilon'+1} \end{pmatrix},$$

$$D' = q^{-jJ'-j'J} \begin{pmatrix} q^{-j-j'} & 0 & 0 & 0 \\ 0 & q^{j-j'} & 0 & 0 \\ 0 & 0 & q^{-j+j'} & 0 \\ 0 & 0 & 0 & q^{j+j'} \end{pmatrix}.$$

Since $a = \mathbf{i}^\alpha = (-1)^{\varepsilon+1} \mathbf{i} q^{-2j}$, the formulas make appear two square roots of a :

$$\sqrt[2]{a} = \mathbf{i}^{\alpha/2} \quad \text{and} \quad \sqrt[2]{a} = \mathbf{i}^{\varepsilon+\frac{3}{2}} q^{-j} = s \sqrt[2]{a}.$$

That way, computing any of the diagonal coefficients of $D^H(D^\sigma)^{-1}$ we find the announced element. 2.7

3. AN INTEGRAL FORM OF $U_q \mathfrak{gl}(n|1)$ AND ITS SPECIALIZATION

3.1. Quasitriangular Hopf superalgebra $U_q \mathfrak{gl}(n|1)$. Here we define the h -adic quasitriangular Hopf superalgebra $U_q \mathfrak{gl}(n|1)$ that we will use to construct the Links-Gould invariant $LG^{n,1}$. The conventions we use for generators and relations are those chosen by Zhang and De Wit in [15, 2]. $\mathcal{I} = \{1, 2, \dots, n+1\}$ will be the set of indices. We introduce a grading $[a] \in \mathbb{Z}/2\mathbb{Z}$ for any $a \in \mathcal{I}$ by setting

$$[a] = 0 \text{ if } a \leq n \text{ and } [a] = 1 \text{ when } a = n+1.$$

The superalgebra has $(n+1)^2$ generators divided into three families. There are $n+1$ even Cartan generators E_a^a . There are $\frac{1}{2}n(n+1)$ lowering generators E_a^b parametrized by $a < b$. Finally there are $\frac{1}{2}n(n+1)$ raising generators E_b^a , with $a < b$. The degree of E_a^b is given by $[a] + [b]$.

For $a \in \mathcal{I}$, $a \neq n+1$, set $K_a = q^{E_a^a}$, and set $K_{n+1} = q^{-E_{n+1}^{n+1}}$. In the following $[X, Y]$ denotes the super commutator $[X, Y] = XY - (-1)^{[X][Y]} YX$.

Now let us present the relations there are between elements of $U_q \mathfrak{gl}(n|1)$.

For any $a, b \in \mathcal{I}$ with $|a - b| \geq 2$ and for any c in the interval between a and b ,

$$E_b^a = E_c^a E_b^c - q^{\text{sign}(a-b)} E_b^c E_c^a.$$

For any $a, b \in \mathcal{I}$,

$$E_a^a E_b^b = E_b^b E_a^a, \quad E_a^a E_{b\pm 1}^b = E_{b\pm 1}^b (E_a^a + \delta_b^a - \delta_{b\pm 1}^a)$$

$$[E_{a+1}^a, E_b^{b+1}] = \delta_b^a \frac{K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1}}{q - q^{-1}}$$

$$\text{which generalizes for } a < b \text{ to } [E_b^a, E_a^b] = \frac{K_a K_b^{-1} - K_a^{-1} K_b}{q - q^{-1}},$$

$$(E_{n+1}^n)^2 = (E_n^{n+1})^2 = 0, \text{ which implies } (E_{n+1}^i)^2 = (E_i^{n+1})^2 = 0 \text{ for } i < n+1.$$

The Serre relations: for any $a, b \in \mathcal{I}$ with $|a - b| \geq 2$,

$$E_{a+1}^{a+1} E_b^{b+1} = E_b^{b+1} E_{a+1}^{a+1}, \quad E_{a+1}^a E_{b+1}^b = E_{b+1}^b E_{a+1}^a,$$

and for $a \leq n-1$,

$$E_{a+1}^a E_{a+2}^a = q E_{a+2}^a E_{a+1}^a, \quad E_{a+1}^{a+1} E_{a+2}^{a+2} = q E_{a+2}^{a+2} E_{a+1}^{a+1},$$

$$E_{a+2}^a E_{a+2}^{a+1} = q E_{a+2}^{a+1} E_{a+2}^a \quad \text{and} \quad E_{a+2}^{a+2} E_{a+1}^{a+2} = q E_{a+1}^{a+2} E_{a+2}^{a+2}.$$

These relations can be completed into a set of “quasi-commutation” relations indexed by pairs of root vectors (see [2, Lemma 1] where a reordering algorithm gives a constructive proof of the Poincaré-Birkhoff-Witt theorem) but these relations are redundant over the field $\mathbb{C}(q)$.

We consider the Hopf algebra structure given by the coproduct

$$\begin{aligned}\Delta(E_{a+1}^a) &= E_{a+1}^a \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{a+1}^a, \quad \Delta(E_a^{a+1}) = K_a^{-1} K_{a+1} \otimes E_a^{a+1} + E_a^{a+1} \otimes 1 \\ \Delta(K_a) &= K_a \otimes K_a \quad \text{and} \quad \Delta(E_a^a) = E_a^a \otimes 1 + 1 \otimes E_a^a\end{aligned}$$

which admits² the universal R -matrix $R^{\mathfrak{gl}} = D^{\mathfrak{gl}} \check{R}^{\mathfrak{gl}}$ with $D^{\mathfrak{gl}} = q^{\sum_{i \leq n} E_i^i \otimes E_i^i - E_{n+1}^{n+1} \otimes E_{n+1}^{n+1}}$ and

$$\check{R}^{\mathfrak{gl}} = \prod_{i=1}^n \left(\prod_{j=i+1}^n e_q((q - q^{-1})E_j^i \otimes E_i^j) \right) e'_q(E_{n+1}^i \otimes E_i^{n+1}),$$

where $e'_q(x) = (1 - (q - q^{-1})x)$, $e_q(x) = \sum_{k=0}^{+\infty} \frac{x^k}{(k)_q!}$, $(k)_q = \frac{1 - q^k}{1 - q}$ and $(k)_q! = (1)_q(2)_q \dots (k)_q$. Remark that the order of the factors matters in $\check{R}^{\mathfrak{gl}}$.

3.2. Integral form and interesting subalgebras. We now give an integral form of $U_q \mathfrak{gl}(n|1)$ which supports evaluation at $q = -1$. Let \mathcal{A}_q be the $\mathbb{Z}[q, q^{-1}]$ -subalgebra of $U_q \mathfrak{gl}(n|1)$ generated by elements K_a , $\mathcal{E}_b^a := (q - q^{-1})E_b^a$ when $a < b$ and $\mathcal{E}_b^a := E_b^a$ when $a > b$. The relations of $U_q \mathfrak{gl}(n|1)$

$$[E_b^a, E_a^b] = \frac{K_a K_b^{-1} - K_a^{-1} K_b}{q - q^{-1}}$$

for $a < b$, are replaced in algebra \mathcal{A}_q by

$$[\mathcal{E}_b^a, \mathcal{E}_a^b] = K_a K_b^{-1} - K_a^{-1} K_b.$$

Still, \mathcal{A}_q admits a presentations similar to that of $U_q \mathfrak{gl}(n|1)$. No additional relations are needed because the analog of the above commutation relations are enough to express any element in the Poincaré-Birkhoff-Witt basis.

In the bosonization \mathcal{A}_q^σ of \mathcal{A}_q , define for $i = 1, \dots, n$ the algebra

$$A_i = \langle e_i = -\mathcal{E}_{n+1}^i \sigma, f_i = \mathcal{E}_i^{n+1}, k_i = K_i K_{n+1}^{-1} \sigma \rangle \subset \mathcal{A}_q^\sigma.$$

Proposition 3.1. *Algebra A_i is isomorphic to U . Indeed:*

$$\begin{aligned}e_i f_i - f_i e_i &= k_i - k_i^{-1}, \\ k_i e_i + e_i k_i &= k_i f_i + f_i k_i = 0.\end{aligned}$$

Proof. Direct computations from the defining relations of \mathcal{A}_q and Lemma 1 of [2]. In particular, $e_i f_i - f_i e_i = -\mathcal{E}_{n+1}^i \sigma \mathcal{E}_i^{n+1} + \mathcal{E}_i^{n+1} \mathcal{E}_{n+1}^i \sigma = [\mathcal{E}_{n+1}^i, \mathcal{E}_i^{n+1}] \sigma = k_i - k_i^{-1}$. 3.1

Remark 3.2. However, A_i is not isomorphic to U as a Hopf algebra (except for A_n), which can be seen by looking at the coproduct of elements of A_i in \mathcal{A}_q . This will not be a problem for us.

Set $1 \leq i \neq j \leq n$. Using [2] Lemma 1 once again, we want to see at what conditions any $x \in A_i$ and $y \in A_j$ commute.

Lemma 3.3. *We have the following commutations:*

$$\begin{aligned}e_i e_j &= -q^{-1} e_j e_i, \quad f_i f_j = -q^{-1} f_j f_i, \quad k_i k_j = k_j k_i, \\ \text{if } i < j, \quad e_i f_j - f_j e_i &= \sigma K_j K_{n+1}^{-1} \mathcal{E}_j^i, \quad \text{otherwise } e_i f_j - f_j e_i = \sigma (q - q^{-1}) \mathcal{E}_j^i K_{n+1} K_i^{-1}, \\ k_j e_i &= -q^{-1} e_i k_j, \quad k_j f_i = -q f_i k_j.\end{aligned}$$

²we use here the coproduct and R -matrix of [7] conjugated by $D^{\mathfrak{gl}}$.

Proof. The first two equalities correspond to [2, Eq. (38) and (39)] and the two brackets $[e_i, f_j]$ correspond to [2, Eq. (36) (c) and (d)]. 3.3

Corollary 3.4. *Setting $q = -1$, in any quotient of \mathcal{A}_{-1}^σ such that for any $1 \leq i < j \leq n$, $\mathcal{E}_j^i = 0$, the elements of two distinct A_i commute.*

3.3. Highest weight representation $V(0^n, \alpha)$. Let $V(0^n, \alpha)$ be the highest weight irreducible 2^n -dimensional representation of $U_q \mathfrak{gl}(n|1)$ of weight $(0^n, \alpha)$, with $\alpha \notin \mathbb{Z}$. So E_i^i is represented by 0, except for E_{n+1}^{n+1} that is represented by α . Set v_0 a highest weight vector in $V(0^n, \alpha)$ and let $V_q(0^n, \alpha) = \mathcal{A}_q v_0$. The Poincaré-Birkhoff-Witt theorem proves that

$$\left(\prod_{i=1}^n f_i^{m_i} v_0 \right)_{m_i \in \{0,1\}}$$

is a basis for vector space $V(0^n, \alpha)$ and for the free $\mathbb{Z}[q, q^{-1}]$ -module $V_q(0^n, \alpha)$. Set $\mathcal{A}_{-1}^\sigma = \mathcal{A}_q^\sigma \otimes_{q=-1} \mathbb{C}$ and $V_{-1}(0^n, \alpha) = V_q(0^n, \alpha) \otimes_{q=-1} \mathbb{C}$

Proposition 3.5. *In the representation $V_{-1}(0^n, \alpha)$, for any $1 \leq i < j \leq n$, $\mathcal{E}_j^i = 0$. So \mathcal{E}_j^i belongs to the kernel I of the representation $\mathcal{A}_{-1}^\sigma \rightarrow \text{End}(V_{-1}(0^n, \alpha))$. As a consequence, the following map is well defined:*

$$\begin{array}{ccc} \Theta : \bigotimes_{i=1}^n A_i & \longrightarrow & \mathcal{A}_{-1}^\sigma / I \\ \otimes_i x_i & \mapsto & \prod_i x_i \end{array}.$$

Proof. We want to show that for any basis vector $v \in V_{-1}(0^n, \alpha)$ and for $1 \leq i < j \leq n$, $\mathcal{E}_j^i v = 0$. We can write $v = f_1^{i_1} \dots f_n^{i_n} v_0$ where $i_k = 0, 1$.

Using [2] Lemma 1 once more, if

$$\begin{array}{ll} c < i \text{ then } & [E_j^i, E_c^{n+1}] = 0 \text{ by [2, Eq. (40)]} \\ c = i \text{ then } & [E_j^i, E_c^{n+1}] = -K_i K_j^{-1} E_j^{n+1} \text{ by [2, Eq. (36)(a)]} \\ i < c < j \text{ then } & [E_j^i, E_c^{n+1}] = -(q - q^{-1}) K_c K_j^{-1} E_c^i E_j^{n+1} \text{ by [2, Eq. (43)(a)]} \\ j \leq c \text{ then } & [E_j^i, E_c^{n+1}] = 0 \text{ by [2, Eq. (37), (40)].} \end{array}$$

In all cases, $[\mathcal{E}_j^i, f_c] = [\mathcal{E}_j^i, \mathcal{E}_c^{n+1}] = (q - q^{-1})[E_j^i, E_c^{n+1}] = 0$ in \mathcal{A}_{-1}^σ . So $\mathcal{E}_j^i v = f_1^{i_1} \dots f_n^{i_n} (\mathcal{E}_j^i v_0)$. But \mathcal{E}_j^i is a raising generator, so $\mathcal{E}_j^i v_0 = 0$. Using Corollary 3.4, for $i \neq j$ A_i and A_j commute in that representation. 3.5

3.4. $\check{R}^{\mathfrak{gl}}$ makes sense when $q = -1$. Here we intend to show that the non diagonal part $\check{R}^{\mathfrak{gl}}$ of the universal R -matrix of $U_q \mathfrak{gl}(n|1)$ supports evaluation at $q = -1$, which is not obvious given the formula defining $\check{R}^{\mathfrak{gl}}$. In the bosonization $U_q \mathfrak{gl}(n|1)^\sigma$, the universal R -matrix is given by

$$(R^{\mathfrak{gl}})^\sigma = D^{\mathfrak{gl}} (\check{R}^{\mathfrak{gl}})^\sigma = D^{\mathfrak{gl}} \prod_{i=1}^n \left(\prod_{j=i+1}^n e_q(\mathcal{E}_j^i \otimes \mathcal{E}_i^j) \right) (1 + e_i \otimes f_i).$$

Proposition 3.6. *For any $1 \leq i < j \leq n$,*

$$(\mathcal{E}_j^i \otimes \mathcal{E}_i^j - 1) V_q(0^n, \alpha) \otimes V_q(0^n, \alpha) \subset (q+1) \mathbb{Z}[q, q^{-1}]_{loc} V_q(0^n, \alpha) \otimes V_q(0^n, \alpha)$$

where $\mathbb{Z}[q, q^{-1}]_{loc}$ is the localization of $\mathbb{Z}[q, q^{-1}]$ at $(q+1)$. Hence $(R^{\mathfrak{gl}})^\sigma$ induces a well defined automorphism of $V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha)$ where the action of $(\check{R}^{\mathfrak{gl}})^\sigma$ is given by

$$(\check{R}^{\mathfrak{gl}})^\sigma = \prod_{i=1}^n (1 + e_i \otimes f_i).$$

Proof. Define $V = \mathbb{Z}[q, q^{-1}]_{loc} V_q(0^n, \alpha) \subset V(0^n, \alpha)$ so that $V_{-1}(0^n, \alpha) \cong V \otimes_{q=-1} \mathbb{C}$. We wish to prove that for $1 \leq i < j \leq n$, in the representation $V \otimes V$, $e_q(\mathcal{E}_j^i \otimes \mathcal{E}_i^j) = 1 \bmod (q+1)$. Set $1 \leq i < j \leq n$. We show by induction on $k \geq 1$, that

$$\frac{(\mathcal{E}_j^i)^k}{(k)_q!} V \subset (q+1)V.$$

For $k = 1$, it follows from $\mathcal{E}_j^i \in I$ (see Proposition 3.5). Now we suppose it holds for any $l \in \{1, \dots, k-1\}$ and since $\frac{(\mathcal{E}_j^i)^k}{(k)_q!} = \frac{(\mathcal{E}_j^i)^{k-1}}{(k-1)_q!} \frac{\mathcal{E}_j^i}{(k)_q}$ it is enough to show that $\frac{\mathcal{E}_j^i}{(k)_q} V \subset V$.

We know that $\mathcal{E}_j^i V \subset (q+1)V$, so $\frac{\mathcal{E}_j^i}{(k)_q} V \subset \frac{q+1}{(k)_q} V$.

If k is even, $(k)_q = (q+1)(\frac{k}{2})_{q^2}$ with $(\frac{k}{2})_{q^2} = \frac{k}{2} \bmod (q+1)$ so $\frac{\mathcal{E}_j^i}{(k)_q} V \subset \frac{1}{(\frac{k}{2})_{q^2}} V = V$.

If k is odd, $(k)_q = 1 \bmod (q+1)$ and therefore $\frac{\mathcal{E}_j^i}{(k)_q} V \subset (q+1)V$. This concludes the proof. 3.6

4. LINKS-GOULD INVARIANTS AND THE CONJECTURE

4.1. Links-Gould invariants $LG^{n,1}$. The Links-Gould invariants $LG^{n,1}$ are framed link invariants obtained by applying the modified (one has to use a modified trace, see [4]) Reshetikhin-Turaev construction to the ribbon Hopf algebras $U_q \mathfrak{gl}(n|1)^\sigma$ we just studied. Like in the Alexander case, the R -matrix can be divided by the value of the twist so that $LG^{n,1}$ becomes an *unframed* link invariant. Note that this definition and Viro's work [14] show that the first LG invariant $LG^{1,1}$ coincides with the Alexander-Conway polynomial Δ .

There are several sets of variables used in papers studying LG invariants. Three of them appear regularly: (t_0, t_1) , (τ, q) and (q^α, q) . Each set can be expressed in terms of the others using the following defining relations:

$$\begin{aligned} t_0 &= q^{-2\alpha}, \quad t_1 = q^{2\alpha+2}, \\ \tau &= t_0^{1/2} = q^{-\alpha}. \end{aligned}$$

In the case of $LG^{2,1}$, variables (t_0, t_1) nicely lead to a symmetric Laurent polynomial that has all sorts of Alexander-type properties [6].

Here we are interested in what happens to $LG^{n,1}$ when you evaluate q at -1 , or in other words when you set $t_0 t_1 = 1$.

4.2. Proof of the conjecture. Our study of ribbon Hopf algebra $U_q \mathfrak{gl}(n|1)^\sigma$ allows us to prove the following, that was conjectured in [3]:

Theorem 4.1. *For any link L in S^3 , $LG^{n,1}(L; \tau, -1) = \Delta_L(\tau^2)^n$. This can be translated in variables (t_0, t_1) :*

$$LG^{n,1}(L; t_0, t_0^{-1}) = \Delta_L(t_0)^n.$$

The rest of the section is devoted to proving this identity. First we identify $V_{-1}(0^n, \alpha)$ as a $\otimes_i A_i$ -module:

Proposition 4.2. *Equipped with the action of $\otimes_i A_i$ induced by $\Theta : \otimes_i A_i \rightarrow \mathcal{A}_{-1}^\sigma / I$, $V_{-1}(0^n, \alpha)$ is isomorphic to the irreducible representation $\otimes_i V^i$ where each V^i is an A_i -module isomorphic to the 2-dimensional U -module $V_{q^{-\alpha}}$.*

Proof. By $\otimes_i V^i$, we mean the representation

$$\otimes_i \rho_i : \otimes_i A_i \rightarrow \otimes_i \text{End}_{\mathbb{C}}(V^i) \cong \text{End}_{\mathbb{C}}(\otimes_i V^i) \text{ where } \rho_i : A_i \rightarrow \text{End}_{\mathbb{C}}(V^i)$$

Set $a = q^{-\alpha}$. For each i , k_i^2 acts by a^2 on $V_{-1}(0^n, \alpha)$. Thus $V_{-1}(0^n, \alpha)$ is a representation of the 8^n -dimensional semi-simple algebra $\bigotimes_{i=1}^j (A_i / (k_i^2 - a^2))$. But for each A_i , v_0 is a highest weight vector of weight a . So it belongs to a summand of the $\bigotimes_{i=1}^j (A_i / (k_i^2 - a^2))$ -module $V_{-1}(0^n, \alpha)$ of the form $\otimes_i V^i$. Comparing the dimensions which are equal to 2^n for both vector spaces, we have that $V_{-1}(0^n, \alpha) \simeq \otimes_i V^i$. 4.2

Now we study the action of the pivotal element of \mathcal{A}_q^σ in the representation at $q = -1$.

Proposition 4.3. *If $K_{2\rho}^\sigma$ is the pivotal element of \mathcal{A}_q^σ , in the representation $V_{-1}(0^n, \alpha)$,*

$$K_{2\rho}^\sigma = \Theta(\otimes_i \phi_i)$$

where $\phi_i = k_i^{-1} \in A_i$.

Proof. The antipode of $U_q \mathfrak{gl}(n|1)$ satisfies $S(E_{i+1}^i) = -E_{i+1}^i K_{i+1} K_i^{-1}$ and $S^2(E_{i+1}^i) = K_i K_{i+1}^{-1} E_{i+1}^i K_{i+1} K_i^{-1} = K_{2\rho} E_{i+1}^i K_{2\rho}^{-1}$. We can write $K_{2\rho}$ in terms of Cartan generators:

$$K_{2\rho} = K_{n+1}^n \prod_{i=1}^n K_i^{n-2i}.$$

Denoting $\langle a|b \rangle := \sum_{i=1}^n a_i b_i - a_{n+1} b_{n+1}$, and ρ the graded half sum of all positive roots, we find:

$$2\rho = \sum_{i=1}^n (n - 2i) \varepsilon_i + n \varepsilon_{n+1},$$

where ε_i is the i^{th} basis vector of \mathbb{C}^{n+1} and we write any vector $x = \sum_{i=1}^{n+1} x_i \varepsilon_i$ in this basis. $K_{2\rho}$ conjugates element $e_i \in A_i$ as follows:

$$\begin{aligned} K_{2\rho} e_i K_{2\rho}^{-1} &= q^{\langle 2\rho | \varepsilon_i - \varepsilon_{n+1} \rangle} e_i \\ &= q^{(n-2i+n)} e_i \\ &= q^{2n-2i} e_i. \end{aligned}$$

So if $q = -1$,

$$\begin{aligned} \sigma K_{2\rho} e_i K_{2\rho}^{-1} \sigma &= -e_i \\ &= \phi_i e_i \phi_i^{-1} \\ &= \Theta(\otimes_j \phi_j) e_i \Theta(\otimes_j \phi_j^{-1}). \end{aligned}$$

Similarly to Proposition 2.6, we therefore can say that in the irreducible $\otimes_i A_i$ -module $V_{-1}(0^n, \alpha)$, $K_{2\rho}^\sigma$ is a scalar multiple of $\Theta(\otimes_j \phi_j)$. We call this element λ . Since the two maps both act by $q^{n\alpha}$ on the highest weight vector, we conclude that $\lambda = 1$. 4.3

Proposition 4.4. *For any $x \in A_i \otimes A_i \subset \mathcal{A}_q \otimes \mathcal{A}_q$, we have*

$$D^{\mathfrak{gl}} x (D^{\mathfrak{gl}})^{-1} = \mathcal{D}(x)$$

where we identified $A_i \otimes A_i \cong U \otimes U$.

Proof. By a direct computation,

$$D^{\mathfrak{gl}} E_{n+1}^j \otimes 1 = E_{n+1}^j \otimes 1 q^{\sum_{i \leq n} ((E_i^i + \delta_j^i) \otimes E_i^i - (E_{n+1}^{n+1} - 1) \otimes E_{n+1}^{n+1})} = E_{n+1}^j \otimes K_j K_{n+1}^{-1} D^{\mathfrak{gl}}$$

Thus $D^{\mathfrak{gl}} e_j \otimes 1 (D^{\mathfrak{gl}})^{-1} = e_j \otimes k_j$. Similarly $D^{\mathfrak{gl}} f_j \otimes 1 (D^{\mathfrak{gl}})^{-1} = f_j \otimes k_j^{-1}$. Finally $k_i \otimes 1$ clearly commutes with $D^{\mathfrak{gl}}$ and we can conclude using $\tau \circ D^{\mathfrak{gl}} = D^{\mathfrak{gl}} \circ \tau$. 4.4

Proof of Theorem 4.1. Let us sum up what we proved up to now to obtain 4.1. Let $V_{-1}(0^n, \alpha) \simeq \bigotimes_{i=1}^n V^i$ be the isomorphic representations of Proposition 4.2. In the following we fix such an isomorphism. Let V_H^i be a $U_1^H \mathfrak{sl}(2)$ -module structure on V^i extending the representation of A_i . We therefore obtain n commuting R-matrices $R^i = D^i \check{R}^i$ in $\text{End}_{\mathbb{C}}(V^i \otimes V^i) \hookrightarrow \text{End}_{\mathbb{C}}(V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha))$, where the explicit inclusion maps are given by $\iota_i : v \otimes w \mapsto (id^{\otimes i-1} \otimes v \otimes id^{\otimes n-i}) \otimes (id^{\otimes i-1} \otimes w \otimes id^{\otimes n-i})$. By Proposition 3.6,

$$\check{R}_{|q=-1}^{\mathfrak{gl}} = \prod_i \iota_i(\check{R}^i) \in \text{End}_{\mathbb{C}}(V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha)),$$

and by Lemma 2.1, Proposition 4.4, and the density Lemma, the conjugation by $\prod_i \iota_i(D^i)$ is equal to the conjugation by $D^{\mathfrak{gl}}$ in $\text{End}_{\mathbb{C}}(V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha))$. Hence the braidings on $(\bigotimes_{i=1}^n V_H^i) \otimes (\bigotimes_{i=1}^n V_H^i)$ and on $V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha)$ are proportional. Now in the process of computing both the Links-Gould invariant and the Alexander polynomial, the R-matrices are rescaled by the inverse of their twist θ^{-1} so that the invariants become framing independent:

$$\text{trace}_2(\theta^{-1}(\text{Id} \otimes \phi) \tau R) = \text{Id}_{V_{-1}(0^n, \alpha)}$$

(here ϕ denotes any of the pivotal structures which are equal by Proposition 4.3). Hence the rescaled R-matrices $R_{|q=-1}^{\mathfrak{gl}} = \prod_i \iota_i(R_{V^i \otimes V^i}^H)$ and $\bigotimes_i R_{V^i \otimes V^i}^H$ are equal up to reordering factors. Finally, for any braid $\beta \in B_\ell$, the associated operators by the Reshetikhin-Turaev construction correspond up to reordering as well:

$$\Psi_{V_{-1}(0^n, \alpha) \otimes \ell}^{\mathfrak{gl}}(\beta) = \left(\Psi_{V_{-\alpha}^{\otimes \ell}}^{U_1^H \mathfrak{sl}(2)}(\beta) \right)^{\otimes n}.$$

At the end, if $\text{trace}_{2,3,\dots,\ell} \left((\text{Id}_{V_{-1}(0^n, \alpha)} \otimes \phi^{\otimes \ell-1}) \circ \Psi_{V_{-1}(0^n, \alpha) \otimes \ell}^{\mathfrak{gl}}(\beta) \right) = d \cdot \text{Id}_{V_{-1}(0^n, \alpha)}$ when $\text{trace}_{2,3,\dots,\ell} \left((\text{Id}_{V_\alpha} \otimes \phi_U^{\otimes \ell-1}) \circ \Psi_{V_\alpha^{\otimes \ell}}^{U_1^H \mathfrak{sl}(2)}(\beta) \right) = c \cdot \text{Id}_{V_\alpha}$, we obtain

$$d = c^n$$

by considering the trace of these two maps. Indeed, the trace is blind to reordering factors. 4.1

Remark 4.5. In [4], the *LG* invariant is extended to a multivariable link invariant $M(L; q, q_1, \dots, q_c)$ for links with $c \geq 2$ ordered components, taking its values in Laurent polynomials $\mathbb{Z}[q^\pm, q_1^\pm, \dots, q_c^\pm]$. It is shown in [5] that

$$LG^{n,1}(\tau, q) = \left(\prod_{i=0}^{n-1} \frac{q^i}{\tau} - \frac{\tau}{q^i} \right) M(L; q, \tau^{-1}, \dots, \tau^{-1}).$$

The proof in this paper should adapt to show that

$$M(L; -1, q_1, \dots, q_c) = \nabla(q_1, \dots, q_c)^n$$

where ∇ is the Conway potential function, a version of the multivariable Alexander polynomial.

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